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Solitons in Seiberg-Witten theory and D-branes in the derived category

Paul S. Aspinwall and Robert L. Karp

Center for Geometry and Theoretical Physics, Box 90318, Duke University
Durham, NC 27708-0318, USA
E-mail: psa@cgtp.duke.edu, karp@chandra.phy.uc.edu

Abstract: We analyze the "geometric engineering" limit of a type II string on a suitable Calabi-Yau threefold to obtain an $N = 2$ pure SU(2) gauge theory. The derived category picture together with Π-stability of B-branes beautifully reproduces the known spectrum of BPS solitons in this case in a very explicit way. Much of the analysis is particularly easy since it can be reduced to questions about the derived category of $\mathbb{P}^1$.

Keywords: Solitons Monopoles and Instantons, D-branes, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Since the work of Seiberg and Witten [1] several programs have emerged allowing one to compute the stable spectrum of BPS solitons at any given point in the moduli space. A knowledge of central charges allows one to compute the location of walls of stability where solitons *might* decay but one must work harder to show that a given soliton really does decay. This was first done in [2], indirectly, by using global properties of the moduli space.

One may compactify a type II string on a Calabi-Yau threefold X to obtain a theory in four dimensions with a nonperturbative nonabelian gauge symmetry. The systematics of obtaining a particular gauge group with a particular matter content are well understood [3]. By going to a particular corner of the moduli space [4] one can then decouple gravity to obtain a supersymmetric Yang-Mills field theory. One says the field theory has been "geometrically engineered" [5] from a type II string theory.

One would then like to obtain the spectrum of BPS solitons of the Yang-Mills theory from a knowledge of the BPS states in the type II string theory — namely BPS D-branes. BPS D-branes occur in two types on a Calabi-Yau threefold. The A-branes may be naïvely pictured as special Lagrangian 3-manifolds in X, while B-branes may be naïvely pictured as holomorphic subspaces of X. The fact that neither of these statements is quite right makes D-branes on Calabi-Yau threefolds a very interesting area of current research.

In [6] (see also [7]) a method of analyzing solitons as A-branes in this context was given. In recent years many other approaches to analyzing the BPS spectrum have appeared, including [8]–[11] to name but a few. In this paper we will directly analyze the soliton spectrum yet another way, by using B-branes, the derived category of coherent sheaves and Π-stability. We justify this in several ways:

- We believe that ultimately this provides a relatively simple way of determining soliton spectra, even though one has to first learn the machinery of the derived category.

- One can view this paper as a test of the idea of Π-stability, which it passes beautifully.

- This is probably the easiest example of Π-stability to understand completely.

- One might argue that the degree of rigour in the B-brane picture compares very favourably with most of the other methods. For example, at this point in time it is probably safe to say that B-branes are better-understood than A-branes, since the topological A-model is subject to world-sheet instanton corrections.

Recent progress in the derived category picture [12]–[16] appears to promise a substantially complete understanding of B-branes in the limit of weak string coupling. In particular it incorporates all α'-corrections. Since we wish to analyze the stability of solitons, the key question for us to analyze will concern the *stability* of objects in the derived category. This is the subject of "Π-stability". Criteria for Π-stability have been discussed in [17]–[20] with further examples discussed in [21, 22]. The technique consists of starting at large radius limit, where one may use the classical picture of B-branes as holomorphic submanifolds, and then following the relative "gradings" of the stable branes as one moves along paths in the moduli space.

This is quite a formidable computation in general, but it has been pursued on some cases (see [23, 20, 21] for example). In the model studied in this paper, the Calabi-Yau threefold is a K3-fibration over a $\mathbb{P}^1$. Much of the analysis can be described using the derived category of $\mathbb{P}^1$, rather than that of the whole Calabi-Yau threefold, which results in a relatively simple determination of the stable spectra.

The derived category can be a technically intimidating subject to those unfamiliar with the language of homological algebra. In this paper we attempt to keep the technical details to a minimum. While this paper was derived as consequence of the results of a previous paper [22], we hope this paper can be understood to a large extent without a knowledge of this earlier work.

We should note that the paper [24] is similar to this paper in many ways. There the BPS spectrum was obtained by using quivers and θ-stability in the same geometric engineering context that we use. Π-stability is a more general picture than θ-stability so this paper can be thought of as a completion of the work in [24] to the whole moduli space.

In section 2 we review the basic setup of the geometrical engineering required to analyze the Seiberg-Witten theory. Section 3 consists of the analysis of the stable spectrum. This analysis is performed for three regions of the moduli space. Firstly, in section 3.2, near the large radius limit. Then we cross the line of marginal stability into the strongly-coupled regime in section 3.3, and then, lastly, we push through back into the weak-coupling region in section 3.4. For this last computation we are required to go beyond the derived category of $\mathbb{P}^1$ as we explain in section 3.5.

2. The model

2.1 Engineering the SU(2)

In this section we quickly review how to obtain a pure SU(2) supersymmetric Yang-Mills theory from a type IIA string compactified on a Calabi-Yau threefold X.

Assume X is a K3 fibration over a base $C \cong \mathbb{P}^1$ with a section. If C (i.e. the section) is large, then we may expect any nonabelian gauge symmetry to arise essentially from the type IIA string compactified on a K3 [25, 26]. One way for this to occur is the appearance of an A-D-E singularity in the K3 fibres. Another way is that the K3 fibres acquire just the right volume. These situations are essentially T-dual to each other (see, for example, [27] for a review).

In the case that singularities appear in the K3 fibres one may focus attention near these singularities and analyze this non-compact region. This is generally the method used in "geometric engineering" [5]. While we could do that here, for simplicity of exposition, we will use the case where the K3 fibres acquire a particular volume to achieve an enhanced gauge symmetry. We will use the well-known example of [28], where X is a degree 8 hypersurface in the resolution of the weighted projective space $\mathbb{P}^4_{\{2,2,2,1,1\}}$.

In order to analyze the quantum-corrected moduli space of complexified Kähler forms on X we require an analysis of the mirror Y. Y is given as a $(\mathbb{Z}_4)^3$ quotient of the same hypersurface with defining equation

$$a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^4 + a_2 z_2^4 + a_3 z_3^4 + a_4 z_4^8 + a_5 z_5^8 + a_6 z_4^4 z_4^4 \,. \tag{2.1}$$

The "algebraic" coordinates on the moduli space of complex structures of Y are then given by

$$x = \frac{a_1 a_2 a_3 a_6}{a_0^4}, \quad y = \frac{a_4 a_5}{a_6^2} \,. \tag{2.2}$$

Now use the mirror map to relate these coordinates to the complexified Kähler form on X. Let $(B + iJ)_x$ represent the component proportional to the area of a curve in the K3 fibre, and let $(B + iJ)_y$ represent the component giving the area of the base C. Then

$$\begin{aligned} (B + iJ)_x &= \frac{1}{2\pi i} \log x + O(x, y) \\ (B + iJ)_y &= \frac{1}{2\pi i} \log y + O(x, y) \,, \end{aligned} \tag{2.3}$$

where the higher-order terms are obtained from the Picard-Fuchs equations in the usual way.

Next we compute the locus of "bad" conformal field theories Δ, which can also be thought of as the locus where Y becomes singular. The "primary" component of Δ can be computed as

$$\Delta_0 = (1 - 2^8 x)^2 - 2^{18} x^2 y \,. \tag{2.4}$$

There is also another component

$$\Delta_1 = 1 - 4y \,, \tag{2.5}$$

with $\Delta = \Delta_0 \Delta_1^3$.

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The enhanced gauge symmetry will not actually appear unless one is at the limit where C becomes infinitely large, i.e. $y = 0$. For such a nonperturbative effect we must lie on the discriminant locus, forcing $x = 2^{-8}$.

To probe the quantum effects of SU(2) Yang-Mills theories we copy the trick from [4] and do a double scaling limit to zoom in on this point $(x, y) = (2^{-8}, 0)$. In particular we set

$$u = \frac{1 - 2^8 x}{2\sqrt{y}}, \tag{2.6}$$

and take the limit $y \to 0$. Δ_0 now intersects the u-plane at $u = \pm 1$. This is the u-plane of Seiberg and Witten [1].

We will now use monodromy to establish which solitons are becoming massless where. Traditionally this is done using matrices denoting the monodromy action on *charges*. We want to know the monodromy action on the solitons themselves rather than just the charges, so we use Fourier-Mukai transforms. In particular we rely heavily on the results and conjectures of [22].

First set $y = 0$ and perform monodromy around $x = 2^{-8}$. As described in the main example of [22], this corresponds to an "EZ-transform" [29] associated to the fibration $\pi : X \to C$. According to [22] *the D-branes on X which become massless at this point are generated by $\mathbf{D}(C)$, the derived category of C*. In particular, the basic objects of interest in $\mathbf{D}(X)$ are $\pi^* \mathsf{z}$ for $\mathsf{z} \in \mathbf{D}(C)$.

This statement is key to this paper. Because of the scaling limit we are taking, any D-brane which is not massless at $(x, y) = (2^{-8}, 0)$ has effectively infinite mass and will be ignored. Thus all the solitons of interest in SU(2) gauge theory are associated to the derived category of $C \cong \mathbb{P}^1$. The simplicity of the soliton spectrum of SU(2) supersymmetric gauge theory is thus related to the fact that the derived category of $\mathbb{P}^1$ is easy to analyze.

One cannot ask for a simpler nontrivial space to analyze as far as the derived category is concerned. In fact, one may build all objects in $\mathbf{D}(C)$ from two basic objects: $\mathscr{O}_C$ — the structure sheaf of C, and $\mathscr{O}_p$ — the skyscraper sheaf of a point $p \in C$. That is, all objects can be constructed by binding together (in a sense to be described) and translating these basic ingredients.

2.2 Identifying the central charges

In order to analyze the stability of solitons we need to find the central charges of the relevant objects in the derived category. These central charges are determined by periods, and so we need to solve the Picard-Fuchs equations. For our Calabi-Yau threefold X the relevant Picard-Fuchs differential operators are (with $\Theta_x = x\partial_x$, and similarly for Θ_y) [28, 30]

$$\begin{aligned} &\Theta_x^2(\Theta_x - 2\Theta_y) - 4(4\Theta_x + 3)(4\Theta_x + 2)(4\Theta_x + 1), \\ &\Theta_y^2 - 4(\Theta_x - 2\Theta_y)(\Theta_x - 2\Theta_y - 1). \end{aligned} \tag{2.7}$$

Following the analysis of [5, 4] we introduce the variables

$$x_1 = \frac{4y}{(2^8 x - 1)^2}, \qquad x_2 = 2^8 x - 1. \tag{2.8}$$

For the field theory limit we scale $x_2 \to 0$, while $x_1 = u^{-2}$ assumes arbitrary values.

In terms of the new variables, the second Picard-Fuchs operator in (2.7) becomes

$$-\frac{1}{4}x_1{x_2}^2\,(x_2+1)^2\,\partial_2^2 - {x_1}^2\left(4x_1{x_2}^2+4x_1x_2+x_1-1\right)\partial_1^2 + \\ +{x_1}^2x_2\,(x_2+1)\,(2x_2+1)\,\partial_1\partial_2 - \frac{1}{2}x_1\left(-2+10x_1x_2+10x_1{x_2}^2+3x_1\right)\partial_1\,. \quad (2.9)$$

For brevity we focus on the semiclassical regime of the gauge theory, i.e. $u = \infty$, although, as pointed out in [5], the monopole and dyon regimes allow for a similar treatment. Accordingly, the solutions we seek are power series expansions around $(x_1, x_2) = (0, 0)$. It turns out that the Picard-Fuchs system (2.7) has four solutions with index $(0, 0)$ and two solutions with index $(0, 1/2)$. Instead of listing all the solutions, we focus on the most interesting one:

$$\sqrt{x_2}\left[1-\frac{1}{16}\,x_1-\frac{15}{1024}\,{x_1}^2-\frac{105}{16384}\,{x_1}^3+O(x_1^4)\right]+x_2^{3/2}\left[84+O(x_1)\right]+O(x_2^{5/2})\,. \quad (2.10)$$

Up to a constant, these are the first few terms of the Seiberg-Witten period a, at least to leading order in x_2. We specialize our analysis for this solution as follows. In the $x_2 \to 0$ limit the leading terms of (2.9) are[1]

$$-\frac{3}{2}x_1\,(x_1-1)\,\partial_1+\frac{1}{4}x_1x_2^2\,\partial_2-\frac{1}{4}x_1x_2^2\,\partial_2^2+\frac{1}{16}-{x_1}^2\,(x_1-1)\,\partial_1^2+{x_1}^2x_2\,\partial_1\partial_2\,, \quad (2.11)$$

and this results in the relevant part of the solution (2.10) satisfying the ODE

$${x_1}^2\,(1-x_1)\,\frac{d^2}{d{x_1}^2}+\frac{3}{2}\,x_1\,(1-x_1)\,\frac{d}{dx_1}+\frac{1}{16}\,. \quad (2.12)$$

Recalling that $x_1 = 1/u^2$, we can make a final change of variables $z = (u-1)/2$, transforming (2.12) into

$$z(1-z)\frac{d^2\Phi}{dz^2}-\frac{\Phi}{4}=0\,. \quad (2.13)$$

The central charge of a D-brane is a function of the D-brane charge. The D-brane charge is measured by K-theory. The K-group of $\mathbb{P}^1$ is $\mathbb{Z}^2$, generated by $H^0(C)$ and $H^2(C)$. We may take the two generators of $\mathbf{D}(C)$, namely $\mathscr{O}_C$ and $\mathscr{O}_p$, to be the generators of the K-group. The Chern characters of the sheaves generate $H^0(C)$ and $H^2(C)$ respectively. We therefore require only two basic central charges

$$\begin{aligned} a &= Z(\pi^*\mathscr{O}_p) \\ a_D &= Z(\pi^*\mathscr{O}_C)\,, \end{aligned} \quad (2.14)$$

to determine the central charges of all solitons. The notation has been chosen to coincide with Seiberg and Witten [1]. The central charges should be ratios of periods, i.e., ratios of solutions to (2.7). In the geometric engineering case, the "denominator" period in these ratios is taken to be the one associated to the 0-brane, which is constant in the scaling

[1]The first operator in (2.7) vanishes identically in this limit.

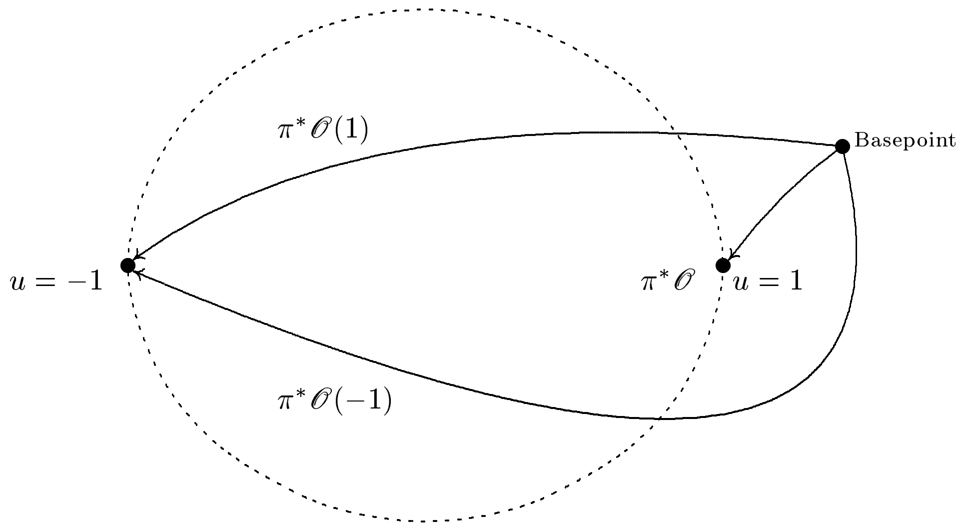

Figure 1: The massless solitons.

limit. This means that the central charges must satisfy (2.13) *up to an additive constant.* This agrees with Seiberg-Witten theory (see [7] for a review and references).

a and a_D may be determined by monodromy as follows. At $u = \pm 1$ the primary component of the discriminant hits the u-plane. As discussed in [22], there is considerable evidence that the structure sheaf $\mathscr{O}_X = \pi^*\mathscr{O}_C$ corresponds to a massless soliton in this case. This statement is basepoint dependent however. What we should really do is choose a basepoint close to, say, $u = 1$ and say that $\mathscr{O}_X$ becomes massless there. Thus $a_D = 0$ at $u = 1$.

Monodromy around $u = -1$ is conjugate, but not isomorphic, to monodromy around $u = 1$. Exactly which D-brane becomes massless depends on the path from the basepoint to $u = -1$. Let us focus on the "direct" paths which do not encircle $u = 1$. Then we have essentially two choices of path which go either side of $u = 1$. As explained in detail in [22], for example, the Fourier-Mukai transform associated to monodromy around $u = -1$ may be derived by composing other Fourier-Mukai transforms. In our case, these rules dictate that $\pi^*\mathscr{O}_C(\pm 1)$ becomes massless at $u = -1$ depending on our choice of path as shown in figure 1. Since $ch(\pi^*\mathscr{O}_C(\pm 1)) = \pi^*(ch(\mathscr{O}_C) \pm ch(\mathscr{O}_p))$, this amounts to $a_D \pm a$ vanishing at $u = -1$. Up to an overall multiplicative constant, this determines a and a_D uniquely to coincide with those given in [1].

This identification with the original work by Seiberg and Witten shows that the structure sheaf $\mathscr{O}_X = \pi^*\mathscr{O}_C$ corresponds to the "magnetic monopole", i.e., the 6-brane wrapping X. The other generator $\pi^*\mathscr{O}_p$ has the right D-brane charge to be associated with the electrically charged "W-boson". In section 3.2 we will see that it really is the W-boson since it is stable. If S is the K3 fibre over the point $p \in C$, then $\pi^*\mathscr{O}_p$ is the structure sheaf of S extended by zero over the rest of X. That is, $\pi^*\mathscr{O}_p$ is a 4-brane wrapped around a K3 fibre. This is consistent with the general assumption (for example in [26]) that the W-bosons arise from D-branes wrapping the K3 fibres (or curves within the K3 fibres).

Classically we have a whole family of W-bosons since we may move p to be any point in C. However, as argued in [26] in this case, quantum mechanics implies that one should count states by computing the cohomology of the moduli space. This means that there is really only one W-boson — not a whole $\mathbb{P}^1$'s worth.

Note that the above analysis shows that the *rank* of a sheaf on $\mathbb{P}^1$ gives the "magnetic charge" while the *degree* of a sheaf gives the "electric charge".

3. The stable spectrum

We are now in a position to establish the spectrum of D-branes at any point in the u-plane. We know that the D-branes in question must be generated by pull-backs to X of objects in the derived category of C. We also know the central charges of such objects. Armed with Π-stability we can solve this problem by determining which objects are stable. Note that the words "generated by" above are a little loaded. We may need to consider morphisms outside $\mathbf{D}(C)$ but we postpone this fact until section 3.4.

3.1 Π-stability

Objects in the derived category are complexes of sheaves. For any such complex A we may define $\mathsf{A}[n]$ simply as the same complex shifted (or "translated") n places left. As explained in [13] a simultaneous shift of all D-branes by n is a gauge symmetry with an odd n representing an exchange of D-branes with anti-D-branes. Relative shifts do matter however. An open string stretched between A and B is only the same as an open string stretched between $\mathsf{A}[n]$ and B if $n = 0$.

In the derived category picture, open strings between a D-brane A and a D-brane B are given by maps in $\mathrm{Hom}(\mathsf{A}[n], \mathsf{B})$ for any value of n. The maps in $\mathrm{Hom}(\mathsf{A}[-1], \mathsf{B})$ are of particular interest as giving these strings a vacuum expectation value forms a "bound" state of A and B as explained in [15]. Mathematically this new object formed by the bound state is called the "Cone" of the corresponding map $f : \mathsf{A}[-1] \to \mathsf{B}$. A nice explanation for this language is given in [31]. In derived category language this leads to a "distinguished triangle":

$$\begin{tikzcd} & \mathrm{Cone}(f) \arrow[dl, "{[1]}"'] & \\ \mathsf{A}[-1] \arrow[rr, "f"] & & \mathsf{B}. \arrow[ul] \end{tikzcd} \tag{3.1}$$

Any vertex of this triangle is a potential bound state of the other two vertices. We refer to [20] for more details. Whether or not each state really is bound or not depends upon the mass of the open string forming the map on the opposite edge of the triangle. Only when this open string is tachyonic do we achieve a bound state. The "[1]" on the arrow in (3.1) means that a left-shift of one is included in the map. This "[1]" may be shuffled around to any edge of the triangle if the objects at the vertices are shifted accordingly. Such shuffles will occur frequently in the computations below.

The rules of computing the masses of these open strings arise from Π-stability as follows [17, 13, 20]. Each stable D-brane is given a "grade" $\varphi \in \mathbb{R}$ which varies continuously

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over the moduli space. It is not single-valued over the moduli space however. It is defined mod 2 by the central charge

$$\varphi(\mathsf{A}) = -\frac{1}{\pi}\arg(Z(\mathsf{A})) \pmod 2\,, \tag{3.2}$$

and $\varphi(\mathsf{A}[n]) = \varphi(\mathsf{A}) + n$. To complete the definition of φ one can declare that the grade of a coherent sheaf lies in some fixed range (say, between -2 and 0) near the large radius limit [13, 32, 20].

The mass of an open string stretched from $\mathsf{A}[-1]$ to B is then proportional to $\varphi(\mathsf{B}) - \varphi(\mathsf{A})$ as explained in [13]. In this way we may determine the regions of stability for all potential D-branes.

Points of marginal stability in the u-plane can appear wherever $\varphi(\mathsf{B}) - \varphi(\mathsf{A}) = 0$. Given (3.2) this means that the ratio of the central charges of A and B is real. For pure SU(2) there is only a two-dimensional lattice of possible D-brane charges, i.e., all central charges are linear combinations of a and a_D. Thus the only possible points of marginal stability occur when a/a_D is real. As is well-known [33, 34], the set of such points forms a near-ellipse passing through $u = \pm 1$. We show this as a dotted line in figure 1 and refer to it as the "ms-line".

3.2 Weak coupling region

Let us begin "outside" the ellipse of marginal stability. Since this is the only ms-line of concern, we may move to very large $|u|$ without effecting the stable spectrum. In terms of X, this allows us to move to the large radius limit of X where all α'-corrections are very small.

In the large radius limit we expect the classical analysis of D-branes such as in [35, 36] to hold true and B-type D-branes should correspond to stable holomorphic vector bundles over complex analytic subspaces of X. In other words, coherent sheaves that are locally free over some complex subspace of X (possibly all of X) and zero elsewhere. Our interest is in sheaves of particular type: those that can be written as the pullback of a sheaf on C described by the previous sentence. As explained earlier, only these sheaves will correspond to solitons associated to the SU(2) theory in question.

The classification of such sheaves on $\mathbb{P}^1$ is very easy. First we have $\mathscr{O}_p$, the skyscraper sheaf of a point. This is classically stable since it has no subsheaves. This pulls back to X to give the structure sheaf of a K3 fibre. As seen in section 2.2, this stable soliton is identified with the W-boson.

The only remaining possibilities are stable vector bundles over $\mathbb{P}^1$. Thanks to a theorem of Grothendieck [37] all vector bundles of rank r over $C \cong \mathbb{P}^1$ are isomorphic to a direct sum of line bundles

$$\mathscr{F} \cong \mathscr{O}_C(s_1) \oplus \mathscr{O}_C(s_2) \oplus \cdots \oplus \mathscr{O}_C(s_r)\,, \tag{3.3}$$

for integers $s_1, s_2, \ldots, s_r$. Such bundles of rank one are stable. To show this we could use classical μ-stability which is related to Π-stability at large radius [17, 20]. We choose to employ Π-stability directly to keep a more consistent language throughout this paper.

Near the large radius limit we use [38, 39]

$$\varphi(\mathscr{F}) = -\frac{1}{\pi}\arg\int_X \exp(B+iJ)\, ch(\mathscr{F})\sqrt{td(X)} + \cdots\,, \tag{3.4}$$

and fix $-2 < \varphi(\mathscr{F}) \leq 0$. It follows that $\varphi(\pi^*\mathscr{O}_p) = -1 + O(\epsilon)$ and $\varphi(\pi^*\mathscr{O}_C(s)) = -3/2 + s\epsilon + \rho(B)\epsilon + O(\epsilon^2)$ where ϵ is a some small positive real number which goes to zero in the large radius limit, and $\rho(B)$ is a B dependent constant. Note that this need not agree with the corresponding values computed by using the a and a_D basis of section 2.2 since the central charge has been left undefined up to an overall constant. What is true however, is that the relative *differences* between grades determined by a and a_D must agree with this large radius limit.

In order to destabilize $\mathscr{O}_C(s)$ we need to build a distinguished triangle with $\mathscr{O}_C(s)$ at one corner. This requires a map $g : \mathscr{E} \to \mathscr{O}_C(s)$ for some stable sheaf $\mathscr{E}$. No map exists for $\mathscr{E} = \mathscr{O}_p$. Suppose $\mathscr{E} = \mathscr{O}_C(r)$ for some r. A map g only exists if $r \leq s$. The case of $r = s$ is trivial as this corresponds to $\mathscr{O}_C(s)$ decaying into itself plus nothing! We may therefore assume $r < s$. As far as sheaves are concerned, distinguished triangles amount to short exact sequences completed into a triangle by adding the arrow with a "[1]" in it. This leads to a triangle

$$\begin{tikzcd} & \mathscr{O}_p^{\oplus(s-r)} \arrow[dl, "{[1]}"'] & \\ \mathscr{O}_C(r) \arrow[rr, "g"] & & \mathscr{O}_C(s), \arrow[ul] \end{tikzcd} \tag{3.5}$$

where, by a slight abuse of notation, $\mathscr{O}_p^{\oplus(s-r)}$ is the skyscraper sheaf of $s-r$ points. But

$$\varphi(\pi^*\mathscr{O}_C(r)) - \varphi(\pi^*\mathscr{O}_p^{\oplus(s-r)}) = -\frac{1}{2} + O(\epsilon)\,, \tag{3.6}$$

which is less than zero showing stability. Thus rank one bundles are stable against decay into other rank one bundles or skyscraper sheaves. We will soon see that higher rank bundles are unstable so we do not need to consider them as possible decay products of rank one bundles.

But wait! The reason higher rank bundles are unstable is because they will be shown to decay into rank one bundles. Such analysis will implicitly assume that the rank one bundles are indeed stable, so we have a circular argument. To be rigorous we should turn to μ-stability. Here a bundle can only be unstable with respect to a subbundle. Clearly a rank one bundle cannot have a higher rank subbundle and so we immediately rule out such decays. What is at work here is that sheaves form an *abelian* category and allow the notion of a subobject. The derived category has no notion of subobject and so Π-stability is always plagued by such circularity. The importance of abelian categories to establish a set of stable solitons at least at some basepoint in the moduli space was emphasized by Douglas [13].

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We now consider the stability of rank 2 bundles. The bundle $\mathscr{O}_C(s) \oplus \mathscr{O}_C(t)$ fits into the triangle

$$\begin{tikzcd} & \mathscr{O}_C(s) \oplus \mathscr{O}_C(t) \arrow[dl] & \\ \mathscr{O}_C(s) \arrow[rr, "f"', "{[1]}"] & & \mathscr{O}_C(t)\,, \arrow[ul] \end{tikzcd} \tag{3.7}$$

where $f \in \mathrm{Hom}(\mathscr{O}_C(s), \mathscr{O}_C(t)[1]) = \mathrm{Ext}^1(\mathscr{O}_C(s), \mathscr{O}_C(t))$. The direct sum $\mathscr{O}_C(s) \oplus \mathscr{O}_C(t)$ occurs when the map f is the zero map. If f is any other map, it will deform the direct sum into some other extension of $\mathscr{O}_C(s)$ by $\mathscr{O}_C(t)$ according to the usual theory of extensions [40]. But $\mathrm{Ext}^1(\mathscr{O}_C(s), \mathscr{O}_C(t))$ has dimension $s-t-1$ if $s \geq t+2$ and is trivial otherwise. Now

$$\varphi(\mathscr{O}_C(t)) - \varphi(\mathscr{O}_C(s)) = \epsilon(t-s) + O(\epsilon^2)\,. \tag{3.8}$$

This implies that if $s < t+2$ then we have no open string along the bottom edge of (3.7) and the direct sum $\mathscr{O}_C(s) \oplus \mathscr{O}_C(t)$ remains as it is, at least according to this triangle. If, on the other hand, $s \geq t+2$, then, according to (3.8), we have a tachyon along the bottom edge of the triangle (3.7). One might therefore be tempted to say that the top vertex of the triangle is stable, but we need to think carefully about this.

The fact that the open string given by f is tachyonic means that it wishes to acquire a nonzero expectation value. That is, we need to "turn on" $f \in \mathrm{Ext}^1(\mathscr{O}_C(s), \mathscr{O}_C(t))$ to make a nontrivial extension. This will deform the direct sum $\mathscr{O}_C(s) \oplus \mathscr{O}_C(t)$ into a rank 2 coherent sheaf, which turns out to be another rank 2 bundle $\mathscr{F}$ [41], which is another extension of $\mathscr{O}_C(s)$ by $\mathscr{O}_C(t)$. But by Grothendieck's theorem, all rank 2 bundles are of the form $\mathscr{F} \cong \mathscr{O}_C(s') \oplus \mathscr{O}_C(t')$, for some integers s' and t'. That is, we have a nontrivial extension

$$0 \to \mathscr{O}_C(t) \to \mathscr{O}_C(s') \oplus \mathscr{O}_C(t') \xrightarrow{h} \mathscr{O}_C(s) \to 0\,. \tag{3.9}$$

By looking at degrees, it is clear that $s+t = s'+t'$, while surjectivity of the map h in (3.9) forces $t < t' \leq s' < s$. Now if $s' \geq t'+2$ then we may repeat the process since we will still have a tachyon. It is clear then that we only achieve stability when $s'-t'$ equals 0 or 1.

Clearly the rôles of s and t can be exchanged in this discussion so we arrive at the conclusion that *the rank two bundle $\mathscr{O}(s) \oplus \mathscr{O}(t)$ will "decay" into $\mathscr{O}(s') \oplus \mathscr{O}(t')$ for $s'+t' = s+t$ and $|s'-t'| \leq 1$.*

So suppose we have a rank two bundle $\mathscr{O}(s) \oplus \mathscr{O}(t)$ with $|s-t| \leq 1$. This is composed of the two "constituents" $\mathscr{O}(s)$ and $\mathscr{O}(t)$. Since $\mathrm{Ext}^1(\mathscr{O}_C(s), \mathscr{O}_C(t)) = \mathrm{Ext}^1(\mathscr{O}_C(t), \mathscr{O}_C(s)) = 0$, there are no open strings to either stabilize or destabilize this combination. We claim therefore that this state is semistable and the two constituents are free to drift apart.[2] In this sense *no* rank 2 bundles form stable bound states.

We show the resulting dyon scattering process in a typical case in figure 2. Following the notation of [1], a sheaf $\mathscr{O}_C(m)$ will have dyon charge $(n_m, n_e) = (1, m)$. Thus two dyons of charge $(1,7)$ and $(1,12)$ will have a tachyonic string between them and will therefore be "attracted" to form a bound state. The resulting dyon will have charge $(2,19)$. This

[2] We do not consider the possibility of marginal bound states here.

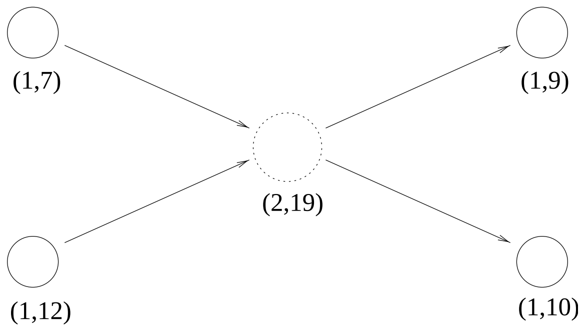

Figure 2: Dyon Scattering.

latter state is composed of dyons of charge $(1, 9)$ and $(1, 10)$ which do not interact by open strings and are free to fly apart.

Clearly we may generalize this result to higher rank. Any higher rank bundle will decay into a set of line bundles whose degrees differ by at most one. This uniquely identifies the decay products of any bundle.

Any dyon of charge (r, m) for $r > 1$ is therefore unstable to decay into dyons with $r = 1$. By a similar reasoning there are no stable states of charge $(0, m)$ for $m > 1$. These would have to correspond to a bound state of skyscrapers $\mathscr{O}_p$ — but clearly these skyscrapers all have the same grade φ and so we could never develop tachyons between them.

The only stable states therefore are $\mathscr{O}_C(m)$ and $\mathscr{O}_p$ which have dyon charge $(1, m)$ and $(0, 1)$ respectively. As always, we also have all the translates given by $\mathscr{O}_C(m)[n]$ and $\mathscr{O}_p[n]$ for any n which includes the corresponding anti-particles. This spectrum is in agreement with known results [1, 2].

3.3 Strong coupling region

Now that we have established a stable set of D-branes near large radius limit, we may venture to cross the ms-line and see what Π-stability tells us. This may be determined purely from the properties of D-branes from the last section, i.e., D-branes stable at large radius.

One might fear that, as we cross the ms-line, a new stable soliton appears that is a bound state of formerly *unstable* solitons. Fortunately, thanks to an analysis of the octahedral axiom in [20], this need not be considered. The formerly unstable solitons must themselves have been some unbound state of stable solitons which may instead be used as the constituent particles for the desired new soliton as we cross the ms-line.

Naturally, what happens as we cross the line of marginal stability depends on the path taken. Let us first assume that we cross the *upper* half of the line of marginal stability by taking the upper path of figure 1.

Integrating the differential equation (2.13) we find that this portion of the ms-line corresponds to a_D/a a real number with $a_D/a = 0$ at $u = 0$ continuously decreasing to $a_D/a = -1$ at $u = -1$.

We now try to define $\arg(a_D/a)$ as a real number varying continuously in the u-plane in order to obtain the φ's of our D-branes. At large radius limit we may use (3.4) to establish that $\arg(a_D/a) = \pi/2$, fixing the $2n\pi$ ambiguity. As we approach the ms-line this number increases steadily to π. That is, slightly above this line of marginal stability $\arg(a_D/a)$ is a little less than π. As we move past the line, $\arg(a_D/a)$ passes through π. To define a continuous function, consistent with the grading, we say that $\arg(a_D/a)$ is a little bit greater than π.

In order to establish the spectrum of stable D-branes as the ms-line is crossed we need to consider all possible distinguished triangles with vertices corresponding to the stable D-branes found in the last section. All of our objects are of the form $\pi^*\mathsf{z}$ for some $\mathsf{z} \in \mathbf{D}(C)$. Unfortunately it is not true that all morphisms $\pi^*\mathsf{x} \to \pi^*\mathsf{y}$ in the derived category of X can be obtained by pulling back a morphism $\mathsf{x} \to \mathsf{y}$ as we discuss in section 3.5. For this section, however, it turns out that we can get away with making this assumption and we need only consider distinguished triangles in the derived category of $\mathbb{P}^1$.

In fact we only need to consider a single distinguished triangle to obtain the result. Namely

$$\begin{tikzcd} & \mathscr{O}_p \arrow[dl, "h"', "{[1]}"] & \\ \mathscr{O}_C(m) \arrow[rr, "f"] & & \mathscr{O}_C(m+1) \arrow[ul, "g"'] \end{tikzcd}, \tag{3.10}$$

with the "[1]" shuffled around as needed. (Actually all distinguished triangles in the derived category of $\mathbb{P}^1$ may be written in terms of (3.10).) We now analyze the masses of the open strings corresponding to the morphisms f, g, and h.

We begin with g. One obtains immediately

$$\begin{aligned} \varphi(\pi^*\mathscr{O}_p[-1]) - \varphi(\pi^*\mathscr{O}_C(m+1)) &= -1 - \frac{1}{\pi}\arg(a) + \frac{1}{\pi}\arg((m+1)a + a_D) \\ &= -1 + \frac{1}{\pi}\arg\left(\frac{a_D}{a} + m + 1\right). \end{aligned} \tag{3.11}$$

Now, since $-1 < a_D/a < 0$, if $m \geq 0$ we have a difference in grading of around -1. Thus the open string is tachyonic and $\mathscr{O}_C(m)$ will not decay into $\mathscr{O}_C(m+1)$ and $\mathscr{O}_p[-1]$. If on the other hand $m < 0$, then the difference in grading will increase through 0 as the ms-line is crossed. Then $\mathscr{O}_C(m)$ will decay into $\mathscr{O}_C(m+1)$ and $\mathscr{O}_p[-1]$.

Thus, for example, $\mathscr{O}_C(-2)$ will decay into $\mathscr{O}_C(-1)$ and $\mathscr{O}_p[-1]$. But $\mathscr{O}_C(-1)$ itself will also decay into $\mathscr{O}_C$ and $\mathscr{O}_p[-1]$. Therefore $\mathscr{O}_C(-2)$ will actually decay into $\mathscr{O}_C$ and two copies of $\mathscr{O}_p[-1]$, assuming that these final products are stable. Anyway, at this point we have shown that all states $\mathscr{O}_C(m)$ for $m < 0$ will decay.

Now consider h. One obtains

$$\varphi(\pi^*\mathscr{O}_C(m)) - \varphi(\pi^*\mathscr{O}_p) = -\frac{1}{\pi}\arg\left(\frac{a_D}{a} + m\right). \tag{3.12}$$

Now if $m \leq 0$, this difference is close to -1 and so we have stability. However, if $m > 0$ then this difference will increase through 0 as we cross the ms-line. Thus $\mathscr{O}_C(m+1)$ will then decay into $\mathscr{O}_C(m)$ and $\mathscr{O}_p$. In other words, all states $\mathscr{O}_C(m)$ for $m > 1$ will decay by this route.

At this point we see that $\mathscr{O}_C(m)$ is only stable if $m = 0$ or 1. Thus to check for stability of $\mathscr{O}_p$ using the map f in (3.10) we need only consider $m = 0$. Then

$$\varphi(\pi^*\mathscr{O}_C(1)) - \varphi(\pi^*\mathscr{O}_C[1]) = -1 - \frac{1}{\pi}\arg\left(\frac{a}{a_D} + 1\right). \tag{3.13}$$

As we cross the ms-line this difference increases through zero. That is, $\mathscr{O}_p$ decays into $\mathscr{O}_C(1)$ and $\mathscr{O}_C[1]$.

The final result is that the only stable solitons after we cross the ms-line are $\mathscr{O}_C$ and $\mathscr{O}_C(1)$ (and their translates corresponding to anti-solitons). Thus, in the example above, $\mathscr{O}_C(-2)$ would actually decay into three copies of $\mathscr{O}_C$ and two copies of $\mathscr{O}_C(1)[-1]$. Note that we have only dyon charges $(1,0)$ and $(1,1)$ in agreement again with known results [1, 2].

We may repeat the above analysis for the *lower* half of the ms-line by taking the lower path of figure 1. The only difference in the analysis is that now $0 < a_D/a < 1$ when we hit the ms-line. The result now is that the only stable solitons consist of $\mathscr{O}_C$ and $\mathscr{O}_C(-1)$. This is in agreement with the fact that monodromy around the monopole point $u = 1$ transforms $\mathscr{O}_C(1)$ into $\mathscr{O}_C(-1)[1]$.

3.4 Returning to weak coupling

Suppose now we continue our journey and pass through the other wall of the region bounded by the ms-line and return to the weakly coupled region as shown by the path in figure 3. One should return to a set of stable solitons physically equivalent to those found in section 3.2, but they may not actually be the same elements of the derived category. That is, we undergo some monodromy in the spectrum.

Monodromy has been analyzed extensively using Fourier-Mukai transforms (see [22] and references therein) but, rather than use this technology, we will just continue the above analysis.

All the distinguished triangles in $\mathbf{D}(C)$ are basically obtained from (3.10). Unfortunately, if we use the analysis of section 3.3, then nothing happens to the spectrum as we cross back into the weakly-coupled region. In particular, the open string corresponding to a map $\mathscr{O}_C \to \mathscr{O}_C(1)$ has mass squared proportional to $\varphi(\pi^*\mathscr{O}_C(1)) - \varphi(\pi^*\mathscr{O}_C[-1])$, which simply gets larger and larger as we follow the path in figure 3. The bound state, which gives $\mathscr{O}_p$, becomes unstable as we enter the strong coupled region and is even more unstable as we leave again.

The problem is that we now need to worry about open strings arising from morphisms that cannot be pulled back from $\mathbf{D}(C)$. In particular, for $m \leq 1$, one can show that

$$H^3(X, \pi^*\mathscr{O}_C(m)) = \mathbb{C}^{1-m}\,, \tag{3.14}$$

where $H^3(X, \pi^*\mathscr{O}_C(m)) \cong \mathrm{Ext}^3(\pi^*\mathscr{O}_C(-m), \pi^*\mathscr{O}_C) \cong \mathrm{Hom}(\pi^*\mathscr{O}_C(-m), \pi^*\mathscr{O}_C[3])$. That is, we have a 2-dimensional Hilbert space of open strings from $\pi^*\mathscr{O}_C(1)[-1]$ to $\pi^*\mathscr{O}_C[2]$. Now, analysis of the periods shows that $\varphi(\pi^*\mathscr{O}_C[2]) - \varphi(\pi^*\mathscr{O}_C(1))$ decreases as we follow the

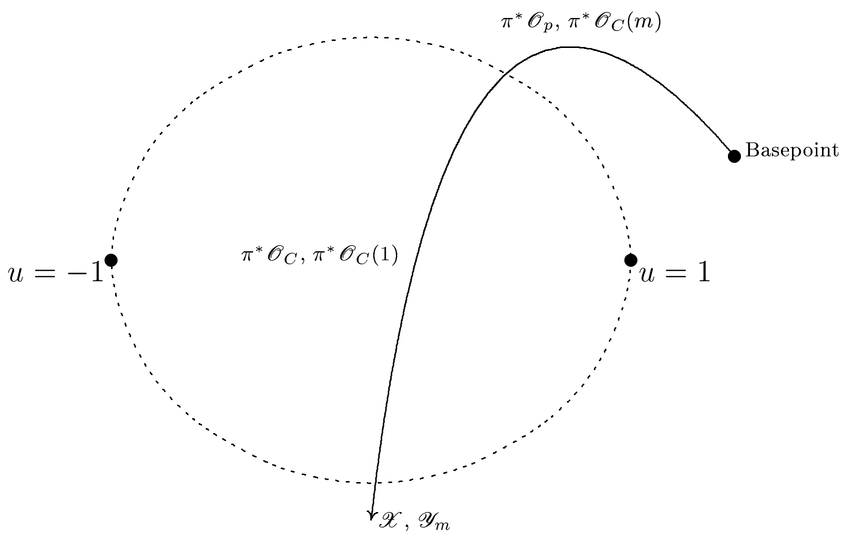

Figure 3: The stable solitons as we follow a path.

path and falls below zero as we hit the lower portion of the ms-line in figure 3. Thus we acquire new bound states of $\pi^*\mathscr{O}_C(1)$ and $\pi^*\mathscr{O}_C[2]$ which we call $\mathscr{X}$. Classically we have a $\mathbb{P}^1$'s worth of these states $\mathscr{X}$ just like the old $\mathscr{O}_p$'s. This is very reminiscent of the 0-branes living on flopped $\mathbb{P}^1$'s analyzed in [42, 21]. $\mathscr{X}$ sits in the following distinguished triangle:

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, "{[1]}"'] & \\ \pi^*\mathscr{O}_C(1)[-1] \arrow[rr] & & \pi^*\mathscr{O}_C[2]. \arrow[ul] \end{tikzcd} \tag{3.15}$$

Consider this to be a special case of the following distinguished triangle

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, "{[1]}"'] & \\ \mathscr{Y}_m \arrow[rr] & & \mathscr{Y}_{m+1}\,, \arrow[ul] \end{tikzcd} \tag{3.16}$$

where $\mathscr{Y}_0 = \pi^*\mathscr{O}_C[2]$ and $\mathscr{Y}_{-1} = \pi^*\mathscr{O}_C(1)[-1]$. This may be used to define $\mathscr{Y}_{m+1}$ inductively as a bound state of $\mathscr{Y}_m$ and $\mathscr{X}$, or $\mathscr{Y}_{m-1}$ as a bound state of $\mathscr{Y}_m$ and $\mathscr{X}[-1]$. This defines (at least up to isomorphism) objects $\mathscr{Y}_m$ in $\mathbf{D}(X)$ for $m = -\infty \ldots \infty$.

Working through the gradings using the rules of [20] it is not hard to show that *all* of the $\mathscr{Y}_m$'s become stable as we pass below the ms-line back into the weak-coupling regime. There are no other homomorphisms between these stable objects of the right relative grades to produce any new stable states. Thus the stable spectrum consists of $\mathscr{X}$ and $\mathscr{Y}_m$ for all m.

Of course these states should correspond to the original set of stable states under monodromy around $u=1$, i.e., the primary component of the discriminant. One can use the transforms discussed in [43] to show that this is indeed the case. It is easy to compute the dyon charges of $\mathscr{X}$ and $\mathscr{Y}_m$ as $(2,1)$ and $(1+2m,m)$ respectively which is in agreement with the known monodromy matrices [1].

It is worth dwelling on a couple of points raised in the analysis of this section. First let us consider the "cascade" of new states coming from (3.16) as we crossed the ms-line.

The rules of computing grades analyzed in [20] said that when a new bound state is formed, the grade of the new state is identical to the grades of the constituent states (shifted by 1 as necessary). Thus if A can bind to B then the grades would imply A can bind to the resulting bound state. Thus A appears able to bind an arbitrary number of times to B. However, this can only happen if open strings exist to produce the necessary bindings. In particular, just because there is a morphism from A to B, there might not be a morphism from A to the bound state of A and B. Then only one copy of A would be able to bind to B. The relationship between the dimensions of the various Hom groups can be computed using long exact sequences associated to the triangles. The magic that happens in (3.16) is that these long exact sequences work out just right to make $\dim\operatorname{Hom}(\mathscr{X},\mathscr{Y}_m[1])$ and $\dim\operatorname{Hom}(\mathscr{Y}_{m+1},\mathscr{X})$ independent of m. It is this property that allows us to add $\mathscr{X}$'s indefinitely to the D-branes $\mathscr{Y}_m$.

The other point is that the original set of states at weak coupling all look nice in the sense that they have clear interpretations in terms of D-branes wrapped on cycles in X, whereas after monodromy we have more exotic objects in the derived category that cannot (except for $\mathscr{Y}_{-1}$ and $\mathscr{Y}_0$) be written as sheaves. Thus if we sit in the strong coupling region there seems to be an asymmetry between moving up or moving down into the weak coupling regime.

However, if we view things purely in the derived category they look more symmetric. When we sit in the strong-coupling region we have only two stable D-branes which we denote A and B. As we move across the ms-line into the weak coupling regime these two solitons can form a bound state W which plays the rôle of the W-boson. Crossing the ms-line upwards this bound state is formed by condensing a tachyon given by the open string associated to the map $\mathsf{A}\to\mathsf{B}$. Crossing the ms-line downwards is exactly the same except this time W is formed from the open string $\mathsf{B}\to\mathsf{A}[3]$. This is essentially Serre duality at work and is very similar to the examples studied in [20] for finding exotic solitons on the quintic threefold. It is also entertaining to note that the "3" appearing here shows that the intrinsic structure of the solitons in Seiberg-Witten theory "knows" that it should be associated with a Calabi-Yau of dimension three!

Once we have crossed into the weak coupling regime, W may bind to an arbitrary number of copies of A or B to form the remaining stable solitons. This description is valid independent of any basepoint chosen. The basepoint is only required to explicitly describe the solitons as complexes of sheaves.

3.5 $\mathbf{D}(C)$ versus $\mathbf{D}(X)$

Finally let us note again that in sections 3.2 and 3.3 we were able to couch our description of the D-branes purely in terms of the derived category of the base $C\cong\mathbb{P}^1$ of the K3-fibration. This explains the relative simplicity of the analysis of solitons in Seiberg-Witten

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theory. The complete description, in terms of $\mathbf{D}(X)$, may add more morphisms than are seen in $\mathbf{D}(C)$ but these extra open strings play no rôle in tachyon condensation near large radius. Once we venture "too far" however by looking at monodromy in section 3.4 we need the full derived category of X to obtain the correct soliton spectrum.

This may be understood schematically as follows. For two objects a and b in $\mathbf{D}(C)$ we have

$$\mathrm{Hom}_{\mathbf{D}(X)}(\pi^*\mathsf{a}, \pi^*\mathsf{b}) = \mathrm{Hom}_{\mathbf{D}(C)}(\mathsf{a}, \pi_*\pi^*\mathsf{b})\,, \tag{3.17}$$

and thus the difference between the open strings in $\mathbf{D}(C)$ and the open strings in $\mathbf{D}(X)$ essentially corresponds to the difference between b and $\pi_*\pi^*\mathsf{b}$.

At the large radius limit we assume that all B-type D-branes correspond to sheaves, i.e., complexes in $\mathbf{D}(X)$ which have only one nonzero entry. We also confine the relative grades to small range $-2 < \varphi \leq 0$. Now $\pi_*\pi^*\mathsf{b}$ is usually a longer complex than the single entry b but in order to "see" the new terms in the complex in terms of tachyon condensation one needs larger relative grades. Thus one needs to travel a large distance from the large radius limit to detect the new terms in $\pi_*\pi^*\mathsf{b}$. This is precisely what happened in section 3.4. The purely $\mathbf{D}(C)$ morphisms gave us "enough" information to cover the whole moduli space once. It was only monodromy issues that showed $\mathbf{D}(C)$'s inadequacies.

It would be interesting to see if the pure $\mathbf{D}(C)$ picture of the solitons in Seiberg-Witten theory persists for more complicated examples of $N = 2$ theories beyond the pure SU(2) theory considered here. Investigations are currently underway in the case of SU(3) and SU(2) with fundamental matter. We hope to report on these in a future publication.

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